TMA4170 Fourier Analysis

Continuous functions and uniform convergence

$f : \mathcal{U} \longrightarrow \mathbb{R}$ (or $\mathbb{C}$)

Sup-norm: $\|f\|_\infty := \sup_{x \in \mathcal{U}} |f(x)|$

$\Rightarrow$ $|f(x)| \leq \|f\|_\infty \quad \forall\, x \in \mathcal{U}$

f bounded if $\|f\|_\infty < \infty$

Uniform convergence: $f_n \xrightarrow{\text{unif.}} f \iff \|f_n - f\|_\infty \longrightarrow 0$

Modulus of continuity: $\omega_f(r) := \sup\{ |f(x) - f(y)| : x, y \in \mathcal{U},\ |x-y| < r \}$

$\Rightarrow \quad |f(x) - f(y)| \leq \omega_f(|x-y|) \quad \forall\, x, y \in \mathcal{U}$

f uniformly continuous $\iff |f(x) - f(y)| \leq \omega_f(|x-y|) \underset{y \to x}{\longrightarrow} 0 \quad \forall\, x \in \mathcal{U}$

$C(K)$ = cont. func'ns on $K \subset \mathbb{R}^d$, bnd. closed (= compact)

(i) $C(K)$, $\|\cdot\|_\infty$ Banach space

(ii) $f \in C(K) \Rightarrow f$ bounded and uniformly continuous

Interchanging limits and integrals

(a) $C([a,b]) \ni f_n \xrightarrow{\text{unif.}} f \implies \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$

(b) $f(t,x),\ \frac{\partial f}{\partial t}(t,x) \in C([c,d]\times[a,b]) \implies \frac{d}{dt}\int_a^b f\,dx = \int_a^b \frac{\partial f}{\partial t}\,dx$

(c) $\sum_{n=-\infty}^{\infty} f_n(x)$ uniformly convergent $\implies \int_a^b \sum_{n=-\infty}^{\infty} f_n(x)\,dx = \sum_{n=-\infty}^{\infty} \int_a^b f_n(x)\,dx$

$f_n \in C([a,b])$

Weierstrass M-test: $\|f_n\|_\infty \le M_n,\ \sum_{n=-\infty}^{\infty} M_n < \infty \implies \sum_{n=-\infty}^{\infty} f_n(x)$ uniformly and absolutely convergent

Riemann integral / integrable functions

Partition P : $a = x_0 < x_1 < x_2 < \dots < x_N = b$

Upper / lower sums:

$$\mathcal{U}(P, f) := \sum_{j=1}^{N} (\sup_{I_j} f) |I_j| , \qquad I_j = [x_{j-1}, x_j], \quad |I_j| = x_j - x_{j-1}$$

$$\mathcal{L}(P, f) := \sum_{j=1}^{N} (\inf_{I_j} f) |I_j|$$

must be =

Riemann integral: $\int_a^b f(x)\, dx := \inf_P \mathcal{U}(P, f) = \sup_P \mathcal{L}(P, f)$

- Exists if and only if f is:

Riemann integrable: $\forall \varepsilon > 0 \ \exists P$ such that $\mathcal{U}(P, f) - \mathcal{L}(P, f) < \varepsilon$

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Riemann integral and L^p-norms

Riemann integrable function:

(i) f, g R.-int., $c \in \mathbb{R}$, $\varphi \in C(\mathbb{R})$ $\Longrightarrow$ $f + cg$, $f \cdot g$, $\varphi(f)$ R.-int.

(ii) cont., p.w. cont., and monotone func'ns are R.-int.

(iii) Not R.-int.: $f = \chi_{\mathbb{Q}}(x)$, $f = x^{-\frac{1}{2}}$

indicator fn $\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$

Step function: $s(x) = \sum_{j=1}^{N} a_j \chi_{[x_{j-1}, x_j)}(x)$

L^p-norm: $\| f \|_{L^p(U)} := \left(\int_U |f(x)|^p dx \right)^{\frac{1}{p}}$, $p \in [1, \infty)$

Approximation:

For any R. int. f on $[a,b]$, $\exists \{ f_n \}_n$ of step / C / C^k func'ns such that: (i) $\| f_n \|_\infty \leq \| f \|_\infty$, (ii) $\| f - f_n \|_p \underset{n \to \infty}{\longrightarrow} 0$ $\forall p \in [1, \infty)$

Lebesgue L^p-spaces

$\mathcal{R}^p(U) := \{ f$ R.-int. on $U : \|f\|_p < \infty \}$

→ $\|\cdot\|_p$ not norm on $\mathcal{R}^p$, $\mathcal{R}^p$ could not be complete, not good!

$L^p(U) := \{ f$ Lebesgue measurable $: \|f\|_p < \infty \}$

→ Banach space <u>if</u> $f \overset{L^p}{=} g \overset{\text{Def.}}{\iff} \{x : f(x) \neq g(x)\}$ has measure 0 (null set)

Hölder inequality: $p, q \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1 \implies \|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q$

Cor.: $1 \leq p \leq q \leq \infty$, U bnd. $\implies$ (a) $\|f\|_p \leq c\|f\|_q$, (b) $L^\infty(U) \subset L^q(U) \subset L^p(U) \subset L^1(U)$

Approximation:

For any $f \in L^p(U)$, $p \in [1, \infty)$, $\exists \{f_n\}_n$ of step / C / C^k func'ns s.t.

$$\|f - f_n\|_p \xrightarrow[n \to \infty]{} 0$$